Nonlinear parametric instability in double-well lattices

Jaroslav Riedel,[1] Boris A. Malomed,[2] and Eva Majerníková[1]

[1]*Department of Theoretical Physics, Palacký University, Tř. 17, Listopadu 50, 77207 Olomouc, Czech Republic*
[2]*Department of Interdisciplinary Studies, Faculty of Engineering, Tel Aviv University, Tel Aviv 69978, Israel*

A possibility of a nonlinear resonant instability of uniform oscillations in dynamical lattices with harmonic intersite coupling and onsite nonlinearity is predicted. Numerical simulations of a lattice with a double-well onsite anharmonic potential confirm the existence of the nonlinear instability with an anomalous value of the corresponding power index, ≈ 1.57, which is intermediate between the values 1 and 2 characterizing the linear and nonlinear (quadratic) instabilities. The anomalous power index may be a result of a competition between the resonant quadratic instability and nonresonant linear instabilities. The observed instability triggers transition of the lattice into a chaotic dynamical state.

Dynamical lattices with onsite nonlinearity and harmonic intersite coupling constitute a vast class of models that have numerous physical application and are an object of great interest in their own right, see, e.g., Ref. [1]. Among these models, the ones with a *double-well* onsite potential [given, e.g., by the expression (12) below] have special importance, as they directly apply to the description of structural transitions in dielectrics, semiconductors, superconductors, and optical lattices (see recent works [2] and references therein), and find other applications [3].

The simplest dynamical state in lattices represents spatially homogeneous oscillations. This state in conservative lattice models is sometimes stable, and sometimes it is subject to linear modulational instabilities initiating a transition to nontrivial dynamics [4]. An objective of the present work is to demonstrate analytically, and verify by direct simulations, that homogeneous oscillatory states in lattices with the double-well anharmonicity may be subject to a specific *nonlinear* instability, which triggers transition of the lattice into a chaotic dynamical state.

The nonlinear parametric instability, which is a subject of this work, is inherently related to the phonon anharmonism in the lattice. The instability is caused by a resonance involving the uniform oscillations of the lattice and a second harmonic of the phonon mode. A possibility of a nonlinear instability of an *intrinsic localized mode* [5] in the lattice due to the phonon anharmonism was first considered in Ref. [6]. An opposite, and more common, type of the resonance, namely, between a strictly linear phonon mode and a higher harmonic of an intrinsic localized mode is well known to give rise to a slow decay of the localized mode into phonons (see recent works [7] and references therein).

The general form of the lattice equation of motion is

$$\ddot{u}_n + f(u_n)u_n = u_{n+1} + u_{n-1} - 2u_n, \qquad (1)$$

where u_n are real dynamical variables on the lattice, the overdot stands for d/dt, $f(u_n)$ is a polynomial function accounting for the onsite nonlinearity (in fact, nonpolynomial functions can be considered too), and the right-hand side of the equation accounts for the intersite harmonic coupling. The linearized version of Eq. (1) gives rise to phonon modes

$$u_n = A\sin(kn - \omega t), \qquad (2)$$

with an arbitrary infinitesimal amplitude A and the dispersion relation $\omega = 2|\sin(k/2)|$ inside the phonon band, $\omega \leq 2$ (by definition, the frequencies are positive).

A homogeneous oscillatory state $U_0(t)$ is a time-periodic solution to the equation

$$\ddot{U}_0 + f(U_0)U_0 = 0 \qquad (3)$$

with a fundamental frequency Ω. The linear stability of the homogeneous state is determined by a linearized equation for small perturbations δu_n, which is produced by the substitution of $u_n = U_0(t) + \delta u_n$ into Eq. (1),

$$\delta \ddot{u}_n + [f'(U_0(t))U_0(t) + f(U_0(t))]\delta u_n$$
$$= \delta u_{n+1} + \delta u_{n-1} - 2\delta u_n. \qquad (4)$$

In the mean-field approximation, one may describe phonon modes of the type (2) on top of the homogeneous oscillations, replacing the coefficient in front of δu_n on the left-hand side of Eq. (4) by its time-average value $\omega_0^2 \equiv \langle f'(U_0(t))U_0(t) + f(U_0(t))\rangle$. The corresponding dispersion relation for the phonon modes acquires a *gap* ω_0, so that

$$\omega^2 = \omega_0^2 + 4\sin^2(k/2), \qquad (5)$$

which gives rise to the phonon band

$$\omega_0^2 \leq \omega^2 \leq 4 + \omega_0^2 \qquad (6)$$

(if $\omega_0^2 < 0$, the homogeneous oscillations are immediately unstable).

Beyond this simple approximation, the Fourier decomposition of the coefficient $f'(U_0(t))U_0(t) + f(U_0(t))$ in Eq. (4) gives rise to parametrically driven terms $\sim \cos(m\Omega t)\cdot \delta u_n$ with all integer values of m. The linear parametric drive resonates with a perturbation frequency ω, i.e., it may give rise to a *resonant* linear instability, under the condition $m\Omega - \omega = \omega$, or

$$\omega = \omega_{\text{res}}^{(\text{lin})} \equiv (m/2)\Omega. \qquad (7)$$

If any resonant frequency $(m/2)\Omega$ gets into the phonon band (6), the homogeneous oscillatory state is expected to be modulationally unstable, otherwise the resonant linear instability does not take place.

In the latter case, it makes sense to seek for nonlinear parametric instabilities, the simplest of which may be generated by a cubic term, or any higher-order one, in the onsite nonlinearity. Indeed, the cubic term generates a nonlinear correction $\sim U_0(t)(\delta u_n)^2$ to Eq. (4), which may be regarded as a parametric drive that can give rise to a *nonlinear* parametric resonance under the condition $m\Omega - 2\omega = \omega$, or

$$\omega = \omega_{\text{res}}^{(\text{nonlin})} \equiv (m/3)\Omega, \qquad (8)$$

where m is an arbitrary integer different from a multiple of 3, cf. Eq. (7) (if m is a multiple of 3, the linear parametric resonance takes place at the same frequency, so that the nonlinear resonance is insignificant).

Of course, this description has a very approximate nature for two reasons. First, the phonon band (6) was defined in the framework of the mean-field approximation, hence one cannot be sure in the accuracy of the predictions based on the comparison of the resonant frequencies with this band. Second, the full lattice model (1) may give rise to other instabilities, which are not related to the parametric resonance. Therefore, the above consideration should only be considered as a qualitative clue, and an actual possibility of dynamical regimes dominated by the nonlinear resonance must be checked by direct simulations.

Continuing the consideration, we note that, if none of the linear-resonance frequencies (7) gets into the renormalized band (6), but a nonlinear-resonance frequency (8) can be found inside the band, an evolution equation for the amplitude of the corresponding resonant-perturbation mode, $\delta u_n = A(t)\cos(\omega_{\text{res}}t)v_n$, with some spatial profile v_n (it may be, for instance, the above-mentioned localized intrinsic mode), has a general form

$$dA/dt = CA^2, \qquad (9)$$

where C is a constant that depends on a particular form of Eq. (1) and the homogeneous solution $U_0(t)$; cf. a similar equation governing the nonlinear instability of the so-called *embedded solitons* [9]. A solution to Eq. (9) is

$$A = A_0/(1 - CA_0 t), \qquad (10)$$

where A_0 is the initial value of the perturbation amplitude. A drastic difference of the perturbation growth law (10) from the exponential growth in the case of the linear instability is that the nonlinear instability is initially growing much slower than an exponential, and a characteristic time scale of the growth, $\sim 1/(CA_0)$, depends on the initial perturbation A_0, while in the case of the exponential growth it is a fixed constant. However, the nonlinear instability is self-accelerating, and, as a manifestation of that, Eq. (10) formally predicts a singularity at $t = 1/(CA_0)$. In reality, of course, the singularity may not occur, as the above approximation, taking into regard the first nonlinear correction to Eq. (4), becomes irrelevant if $A(t)$ is too large. A natural

conjecture, that will be corroborated by direct simulations below, is that the nonlinear instability leads to a chaotic dynamical state.

It is relevant to mention that, although nonlinear instabilities are less common than the usual linear instability, they occur and play an important role in many physical problems, as diverse as optical solitons in media with competing quadratic and cubic nonlinearities [8], Bose gases, plasma turbulence [9], contact lines in flows, etc. [10]. In this work, we will check the possibility of the nonlinear instability of the homogeneous oscillations in the lattice model (1) with

$$f(u_n) = -u_n^2 + \nu u_n^4, \quad \nu > 0, \qquad (11)$$

which corresponds to the double-well onsite anharmonic potential,

$$V(u_n) = -u_n^4/4 + \nu u_n^6/6. \qquad (12)$$

In this case, Eq. (3) can be solved in terms of elliptic functions, but an explicit result is very cumbersome.

As a typical example, we take homogeneous oscillations produced by Eq. (3) with $\nu = 0.01$ and initial conditions $U_0(0) = 1$ and $\dot{U}_0(0) = 0$. The variable U_0 then performs strongly anharmonic oscillations between the values $(U_0)_{\min} = 1$ and $(U_0)_{\max} = 12.247$ at the fundamental frequency $\Omega = 1.694$, and the gap in the renormalized phonon spectrum (5) is calculated to be $\omega_0 = 3.518$, so that the renormalized band (6) is, in the mean-field approximation,

$$3.518 < \omega < 4.047. \qquad (13)$$

Then, it is straightforward to check that all the linear-resonance frequencies (7) do *not* get into this band (the band as whole fits between the linear resonant frequencies 3.392 and 4.240, which correspond to $m = 4$ and $m = 5$). On the other hand, the nonlinear-resonance frequency (8) corresponding to $m = 7$ is $\omega_{\text{res}}^{(\text{nonlin})} = 3.9573$, which lies *inside* the band (13) [all the other frequencies given by Eq. (13) are located outside the band].

To directly test the instability, small perturbations of the form

$$\delta u_n(0) = A_0\cos(2\pi p_0 n/N), \qquad (14)$$

where N is the net number of sites in the lattice and p_0 is an integer, were added to the homogeneous oscillatory state. The lattice equations of motion were solved for $N = 1000$ and periodic boundary conditions by means of the eighth order explicit Runge-Kutta scheme with a stepsize control such that the time step was dynamically changed within the range $0.05 - 0.3$. It was checked that the relative (per site) error at each step did not exceed 10^{-10}.

The simulations were performed for the perturbations (14) with p_0 taking values in the interval $1 \leq p_0 \leq 30$. In all the cases considered, results were quite similar. Here, we demonstrate a typical example with $p_0 = 20$. Long-time evolution initiated by the small perturbation (14) with $A_0 = 0.05$ is displayed in Fig. 1 in the form of a set of plots showing the
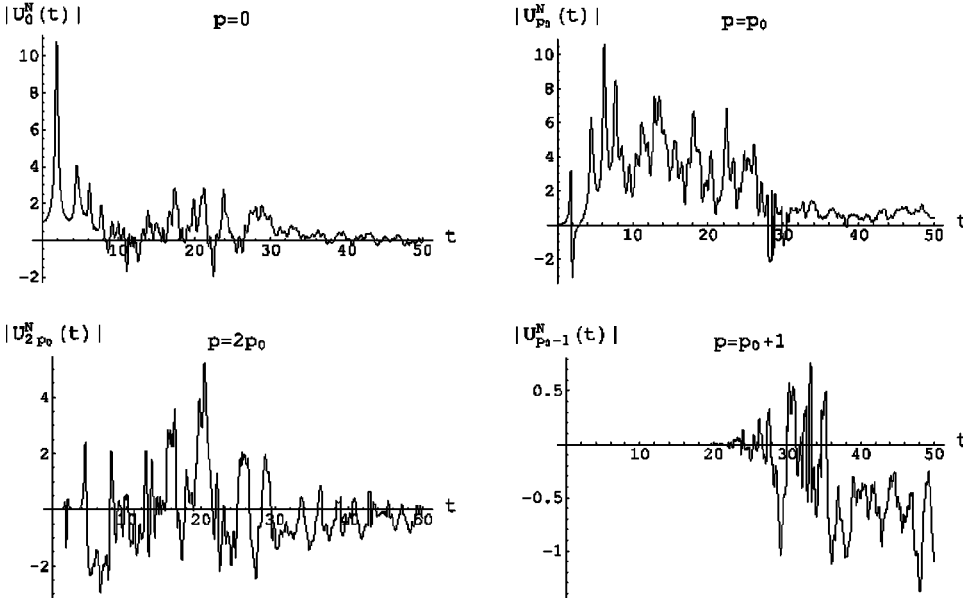
$p_0 = 20, u_0 = 1.0$

FIG. 1. The time dependence for selected Fourier amplitudes $|U_p(t)|$, defined as per Eq. (15). The results are shown for $p=0$, $p=p_0$, $p=2p_0$, and $p=p_0+1$, where $p_0=20$. The lattice size is $N=1000$ (with periodic boundary conditions), and the initial amplitude of the perturbation is $A_0 = 0.05$.

temporal development of several components in the Fourier transform of the lattice field, which are defined as follows:

$$U_p(t) = (2/N) \sum_{n=1}^{N} u_n(t) \exp(2i\pi pn/N), \, p \neq 0;$$

$$U_0(t) = (1/N) \sum_{n=1}^{N} u_n(t). \quad (15)$$

It is obvious that the small perturbation triggers a transition of the lattice into a chaotic state. Fully developed chaos, i.e., a state in which all the lattice modes are involved into the chaotic motion, is attained at $t \approx 22$, when the phonon mode with $p=p_0+1$ gets chaotically excited too, see Fig. 1. To further illustrate the transition to chaos, in Fig. 2 we additionally show in detail, on the logarithmic scale, the growth

of the amplitude $|U_{p_0+1}(t)|$. More detailed studies of the established chaotic state may be of interest in their own right, but this problem is beyond the scope of the present work.

As concerns the nonlinear character of the instability, a crucial issue is the growth of the perturbation at the initial stage. It is necessary to check whether it is indeed essentially different from the familiar exponential law, being, instead, close to the Eq. (10). To this end, in Fig. 3 we display the best fit of the time evolution of the numerically computed Fourier amplitude $|U_{p_0}(t)|$ to a function

$$A_{\text{fit}}(t) = \Delta(1 - \gamma t)^{-\alpha}, \quad (16)$$

where the parameters are found to be $\Delta = 0.041$, $\gamma = 0.560$, and $\alpha = 1.750$.
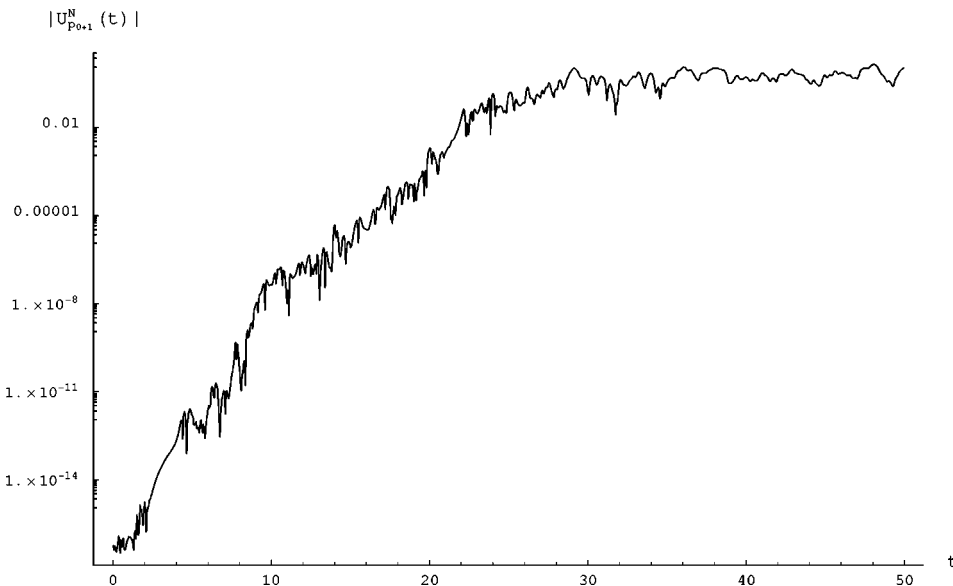

FIG. 2. Details of the evolution of the Fourier amplitude $|U_{p_0+1}(t)|$, shown on the logarithmic scale.
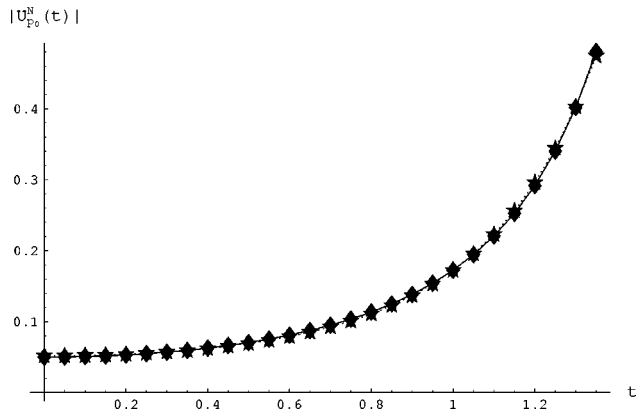
FIG. 3. Fitting the time dependence of the amplitude $|U_{p_0}(t)|$ to the function (16) with $\Delta = 0.0411$, $\gamma = 0.560$, and $\alpha = 1.750$. Diamonds stand for numerical data, and stars (which almost completely overlap with the diamonds) show the closest values provided by the fitting function.

Comparison of these results with Eq. (10) shows a difference in the (most essential) power parameter α. Note that the expression (16) with the empirically found value $\alpha = 1.750$ formally corresponds to a solution to the nonlinear evolution equation $dA/dt = CA^{\beta}$, with an anomalous value of the power index, $\beta \equiv 1 + \alpha^{-1} \approx 1.5714$, that should be compared to Eq. (9), valid in case of the ordinary nonlinear instability [9]. This anomalous value is sort of intermediate between $\beta = 1$ and $\beta = 2$, which are expected for the linear and nonlinear instabilities, respectively. This result may suggest that,

in fact, in the present model we have a competition between the resonant nonlinear instability, qualitatively considered above, and linear instabilities against *nonresonant* perturbations, which were not taken into regard in the above consideration. While an accurate analysis of the full linear stability problem of the homogeneous oscillations is a technically complex problem, that we do not aim to consider here, Fig. 3 clearly shows that the resonant nonlinear instability dominates in the growth of the perturbations.

In conclusion, we have proposed a possibility of a nonlinear resonant instability of homogeneous oscillations in harmonically coupled nonlinear lattices, which is expected to play a dominant role, provided that no resonant frequency accounting for the linear parametric resonant instability gets into the renormalized phonon band, while a frequency that gives rise to a quadratic parametric resonance is found in the band. Numerical simulations of the lattice with a double-well onsite anharmonic potential confirm the existence of nonlinear instability with an anomalous value of the power index ≈ 1.57, which is intermediate between the values 1 and 2, characteristic of the linear and nonlinear instabilities. The onset of the nonlinear instability triggers transition of the lattice into a chaotic dynamical state.

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